Bending of a Uniformly Loaded Annular Plate with Mixed Boundary Conditions

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This paper deals with the analytical solution of a uniformly loaded annular plate in which the inner edge is free and the outer edge restricted by a system of simple and fixed supports. The boundary conditions of the annular plate are thus mixed between the simple and fixed supports. The solution is set up by using the deflection equation and the mixed boundary conditions are written in the form of dual-series equations. By choosing the proper finite integral transform, the dual-series equations can be written in the form of an inhomogeneous Fredholm integral equation. This equation, with a numerical technique, is then reduced to a set of simultaneous equations suitable for numerical solution. The deflections, bending moments, and moment concentration of the annular plate are calculated.

Nomenclature

a	= outer radius of annular plate
b	= inner radius of annular plate
D	= flexural rigidity of annular plate, $Eh^3/12(1-\nu^2)$
h	= plate thickness
E	= Young's modulus
$F(\ )$	= complete elliptic integral of the first kind
$H(\ )$	= Heaviside function
I_n	= modified Bessel function of the first kind and nth order
J_n	= Bessel function of the first kind and nth order
$K(\ )$	= kernel of integral equation
M_r, M_θ	= moments per unit length of plate
q	= uniform load on annular plate
r	= radial coordinate of plate
s,t,u	= dummy variables
V_r	= Kirchhoff shear per unit length of plate
w	= deflection of plate
α	= half-length of simple support
θ	= circular coordinate of annular plate
ν	= Poisson's ratio, assumed to be 0.3
ρ,ξ	= dummy variables
$\Phi(t)$	= auxiliary functions, $\Phi(\alpha\rho)=\Psi(\rho)$ and $\Phi(\alpha\xi)=\Psi(\xi)$
∇^4	= biharmonic operator

Introduction

PROBLEMS of circular plates with the combination of clamped, simply supported, and free boundary conditions have been investigated by many researchers.[1-5] The solutions were focused on vibration, buckling, and bending of the circular plates. Stahl and Keer[6] treated the bending of circular plates; two cases of boundary conditions were considered, namely, clamped/simply supported and simply supported free. Dual-series equations were used for the problem formulation and reduced to a Fredholm integral equation of the second kind.

The bending problems of annular plates under various types of loading are of interest and have many applications in structural engineering. Closed-form solutions can be found for the case of regular boundary conditions, but not for the case of mixed boundary conditions. Therefore, the problems of annular plates with the combination of clamped and simply supported parts (a certain portion of the plate is clamped with the remainder simply supported) under uniformly distributed loads are treated in the present work. The formulation of the problem is in the form of dual-series equations and reduced to the Fredholm integral equation of the second kind as in Ref. 6. The small-deflection theory of thin plates with the notation given in Ref. 9 is used and the deflection of the plate bending is governed by the differential equation

$$D\nabla^4 w = q \tag{1}$$

The general solution that satisfies the two-fold symmetry of the plate shown in Fig. 1 is

$$w = (qr^4/64D) + A_0 + B_0 r^2 + C_0 \log(r/a) + E_0 r^2 \log(r/a) + \sum_{m=2,4,\ldots}^{\infty} [A_m r^m + B_m r^{(m+2)} + C_m r^{-m} + D_m r^{(2-m)}]\cos m\theta \tag{2}$$

in which A_0, B_0, C_0, E_0, A_m, B_m, C_m, and D_m are the unknown constants to be determined by the boundary conditions. It should be noted that, due to the symmetry, only the first quadrant of the annular plate needs to be considered. The boundary conditions on the outer and inner edges are:

Clamped simply at the outer radius $r=a$:

$$w=0, \qquad 0\le\theta\le\frac{\pi}{2} \tag{3}$$

$$\frac{\partial w}{\partial r}=0, \qquad \alpha<\theta\le\frac{\pi}{2} \tag{4}$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] = 0 \qquad 0\le\theta<\alpha \tag{5}$$

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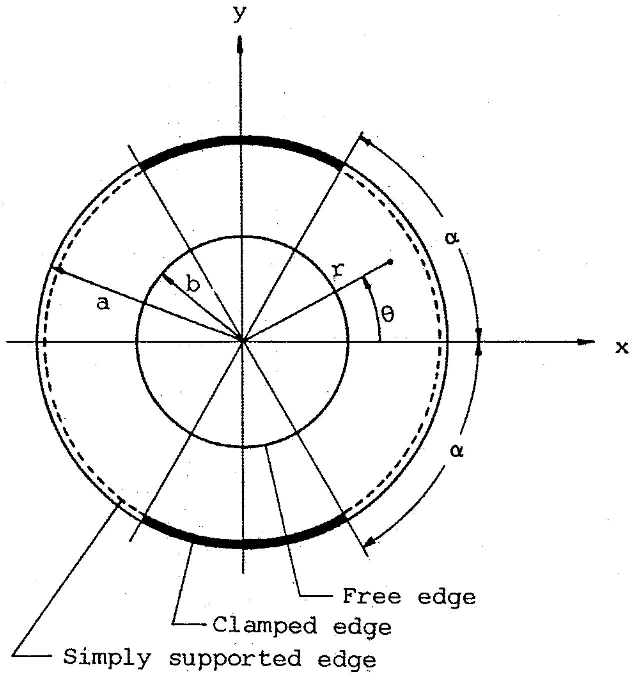

Fig. 1 Annular plate geometry.

Free at the inner radius $r=b$:

$$M_r=-D\left[\frac{\partial^2 w}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]=0$$

$$0\le\theta\le\pi/2 \tag{6}$$

$$V_r=-D\left[\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)+\frac{(1-\nu)}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2}\right)\right]=0$$

$$0\le\theta\le\pi/2 \tag{7}$$

It is seen that the boundary conditions in Eqs. (4) and (5) are mixed.

Substituting Eq. (2) into Eq. (3) yields the relations,

$$A_0=-(qa^4/64D)-B_0a^2 \tag{8}$$

$$A_m=-[B_ma^2+C_ma^{-2m}+D_ma^{2(1-m)}] \tag{9}$$

Substituting Eq. (2) into Eq. (6) and in view of Eq. (8) leads to

$$C_0=\frac{b^2}{(1+\nu)}\left\{\frac{(3+\nu)}{16D}qb^2+2(1+\nu)B_0+\left[2(1+\nu)\log\left(\frac{b}{a}\right)+3+\nu\right]E_0\right\} \tag{10}$$

and

$$m(m-1)(1-\nu)b^{m-2}A_m+(m+1)[(m+2)-\nu(m-2)]b^mB_m+m(m+1)(1-\nu)b^{-(m+2)}C_m-(m-1)[(2-m)+\nu(2+m)]b^{-m}D_m=0 \tag{11}$$

In the same manner, the boundary condition of Eq. (7) leads to the relations

$$E_0=-qb^2/8D \tag{12}$$

and

$$-m^2(m-1)(1-\nu)b^{(m-3)}A_m+m(m+1)[4-m(1-\nu)]\times b^{(m-1)}B_m+m^2(m+1)(1-\nu)b^{-(m+3)}C_m+m(m-1)[4+m(1-\nu)]b^{-(m+1)}D_m=0 \tag{13}$$

The constants C_m and D_m in Eqs. (11) and (13) are written in terms of A_m and B_m as

$$C_m=-\frac{(m-1)(1-\nu)}{(3+\nu)}b^{2m}A_m-\frac{(m^2-1)(1-\nu)^2+(3+\nu)^2}{m(3+\nu)(1-\nu)}b^{2(m+1)}B_m \tag{14}$$

$$D_m=\frac{m(1-\nu)}{(3+\nu)}b^{2(m-1)}A_m+\frac{(m+1)(1-\nu)}{(3+\nu)}b^{2m}B_m \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (9) yields

$$A_m=h_ma^2B_m \tag{16}$$

in which

$$h_m=-\left[1-\frac{(m^2-1)(1-\nu)^2+(3+\nu)^2}{m(3+\nu)(1-\nu)}\left(\frac{b}{a}\right)^{2(m+1)}+\frac{(m+1)(1-\nu)}{(3+\nu)}\left(\frac{b}{a}\right)^{2m}\right]\Big/\left[1-\frac{(m-1)(1-\nu)}{(3+\nu)}\left(\frac{b}{a}\right)^{2m}+\frac{m(1-\nu)}{(3+\nu)}\left(\frac{b}{a}\right)^{2(m+1)}\right] \tag{17}$$

In view of Eqs. (8), (10), and (12), Eq. (2) is written in the form

$$w=\frac{q(r^4-a^4)}{64D}-\frac{4(1+\nu)\log(b/a)+(3+\nu)}{16(1-\nu)D}\times qb^4\log\left(\frac{r}{a}\right)-B_0\left[a^2-r^2-2b^2\frac{(1+\nu)}{(1-\nu)}\log\left(\frac{r}{a}\right)\right]-\frac{qb^2r^2}{8D}\log\left(\frac{r}{a}\right)+\sum_{m=2,4,\ldots}^{\infty}(A_mr^m+B_mr^{m+2}+C_mr^{-m}+D_mr^{2-m})\cos m\theta \tag{18}$$

Substituting Eq. (18) into Eq. (4) and in view of Eqs. (14-16) leads to the relation

$$\frac{1-2(b/a)^2}{16}-\frac{4(1+\nu)\log(b/a)+(3+\nu)}{16(1-\nu)}\left(\frac{b}{a}\right)^4+\frac{2B_0D}{qa^2}\frac{(1-\nu)+(1+\nu)(b/a)^2}{(1-\nu)}+\sum_{m=2,4,\ldots}^{\infty}P_m\cos m\theta=0,\qquad \alpha<\theta\le\pi/2 \tag{19}$$

in which

$$B_m = \frac{a^{2-m}}{G_m D}\, qP_m \qquad (20)$$

$$G_m = mh_m + m + 2 + \frac{m(1-\nu)}{(3+\nu)}\, h_m \left(\frac{b}{a}\right)^{2m} \times \left[(m-1) + (2-m)\left(\frac{b}{a}\right)^{-2}\right] + \frac{(m^2-1)(1-\nu)^2 + (3+\nu)^2}{(3+\nu)(1-\nu)} \times \left(\frac{b}{a}\right)^{2(m+1)} + \frac{(m+1)(2-m)(1-\nu)}{(3+\nu)} \left(\frac{b}{a}\right)^{2m} \qquad (21)$$

Substituting Eq. (18) into Eq. (5) and using the relations in Eqs. (14–16), (20), and (21), the condition in Eq. (5) is written in the form

$$\sum_{m=2,4,\ldots}^{\infty} 2m\left[1 + \frac{(1+\nu)}{2m} + \frac{L_m}{2m}\right] P_m \cos m\theta = \frac{2B_0 D}{a^2}(1+\nu)\left[\left(\frac{b}{a}\right)^2 - 1\right] + \frac{(3+\nu)(b/a)^2}{8} - \frac{(3+\nu)}{16} - \left[4(1+\nu)\log\left(\frac{b}{a}\right) + (3+\nu)\right]\frac{(b/a)^4}{16}$$
$$0 \le \theta < \alpha \qquad (22)$$

in which

$$F_m = m(m-1)(1-\nu)h_m + (m+1)[(m+2) - \nu(m-2)] - m(m^2-1)(1-\nu)^2 h_m \left(\frac{b}{a}\right)^{2m} - \frac{(m+1)[(m-1)(1-\nu)^2 + (3+\nu)^2]}{(3+\nu)}\left(\frac{b}{a}\right)^{2(m+1)} + \frac{m(1-\nu)(1-m)[(2-m)+\nu(2+m)]}{(3+\nu)}\, h_m \left(\frac{b}{a}\right)^{2(m-1)} - \frac{(m^2-1)(1-\nu)[(2-m)+\nu(2+m)]}{(3+\nu)}\left(\frac{b}{a}\right)^{2m} \qquad (23)$$

$$L_m = \frac{F_m}{G_m} - 2m - 1 - \nu \qquad (24)$$

Equations (19) and (22) are dual-series equations and the solution can be found by choosing

$$P_0 = \frac{1}{2}\int_0^\alpha \Phi(t)\,dt, \quad P_m = \int_0^\alpha \Phi(t) J_0(mt)\,dt$$
$$m = 2,4,\ldots \qquad (25)$$

in which t is a dummy variable.

The choice of Eq. (25) has to provide a square root moment singularity at the tips of the clamped segments.[8] Substituting Eq. (25) into Eq. (19) and using the indentity[6]

$$\sum_{m=2,4,\ldots}^{\infty} J_0(mt)\cos m\theta = \frac{1}{2}[(t^2-\theta^2)^{-1/2} H(t-\theta) - 1], \qquad \theta + t < \pi \qquad (26)$$

Eq. (19) becomes

$$B_0 = \frac{qa^2}{2D}\,\frac{(1-\nu)}{(1-\nu)+(1+\nu)(b/a)^2}\left[\frac{1}{2}\int_0^\alpha \Phi(t)\,dt - \frac{1-2(b/a)^2}{16} + \frac{4(1+\nu)\log(b/a) + (3+\nu)}{16(1-\nu)}\left(\frac{b}{a}\right)^4\right] \qquad (27)$$

Substituting Eq. (27) into Eq. (22) and rewriting in the form

$$\frac{d}{d\theta}\sum_{m=2,4,\ldots}^{\infty} P_m \sin m\theta + \frac{(1+\nu)}{2}\sum_{m=2,4,\ldots}^{\infty} P_m \cos m\theta + \frac{1}{2}\sum_{m=2,4,\ldots}^{\infty} L_m P_m \cos m\theta - \frac{(1-\nu)(1+\nu)[(b/a)^2-1]}{4[(1-\nu)+(1+\nu)(b/a)^2]} \times \int_0^\alpha \Phi(t)\,dt = \{(1+3\nu)(b/a)^2 - (1-\nu) - 4(b/a)^2 \times [\nu + (1+\nu)(b/a)^2\log(b/a)]\}/\{16[(1-\nu) + (1+\nu)(b/a)^2]\}, \qquad 0 \le \theta < \alpha \qquad (28)$$

using the identity in Eq. (26) and the identity[6]

$$\sum_{m=2,4,\ldots}^{\infty} J_0(mt)\sin m\theta = \frac{1}{2}(\theta^2 - t^2)^{-1/2} H(\theta - t) - \int_0^\infty [\exp(\pi s) - 1]^{-1} I_0(ts)\sinh(\theta s)\,ds, \qquad \theta + t < \pi \qquad (29)$$

and integrating Eq. (28) once between 0 and θ, Eq. (28) becomes

$$\int_0^\theta \Phi(t)(\theta^2 - t^2)^{-1/2}\,dt = \frac{(1+\nu)(b/a)^2}{2[(1-\nu)+(1+\nu)(b/a)^2]} \times \int_0^\theta\int_0^\alpha \Phi(t)\,dt\,d\theta - \frac{(1+\nu)}{2}\int_0^\theta\int_0^\alpha \Phi(t)(t^2 - s^2)^{-1/2}\,dt\,ds + 2\int_0^\alpha \Phi(t)\int_0^\infty [\exp(\pi s) - 1]^{-1} I_0(ts)\sinh(\theta s)\,ds\,dt - \int_0^\theta\int_0^\alpha \Phi(t)\sum_{m=2,4,\ldots}^{\infty} L_m J_0(mt)\cos m\theta\,dt\,ds + 2\theta\{(1+3\nu)(b/a)^4 - (1-\nu) - 4(b/a)^2[\nu + (1+\nu) \times (b/a)^2\log(b/a)]\}/\{16[1-\nu) + (1+\nu)(b/a)^2]\}$$
$$0 \le \theta < \alpha \qquad (30)$$

Equation (30) is in the form of Abel's integral equation

$$h(\theta) = \int_0^\theta \Phi(t)(\theta^2 - t^2)^{-1/2}\,dt, \qquad 0 \le \theta < \alpha \qquad (31)$$

which has the solution

$$\Phi(u) = \frac{2}{\pi}\,\frac{d}{du}\int_0^u (u^2 - \theta^2)^{-1/2} h(\theta)\,d\theta$$
$$0 < u < \alpha \qquad (32)$$

After some manipulations, using some identities given in Ref. 10 and introducing the new dummy variables $t=\alpha\xi$ and $u=\alpha\rho$, Eq. (30) can be reduced to the form of Fredholm integral equation of the second kind,

$$\Psi(\rho)+\int_0^1 K(\rho,\xi)\Psi(\xi)\mathrm{d}\xi=f(\rho), \qquad 0\leq\rho\leq 1 \tag{33}$$

in which

$$\Psi(\rho)=\Phi(\alpha\rho), \qquad \Psi(\xi)=\Phi(\alpha\xi) \tag{34}$$

$$K(\rho,\xi)=\frac{(1+\nu)}{\pi}\alpha\begin{Bmatrix}(\rho/\xi)F(\rho/\xi), & \xi>\rho\\ F(\xi/\rho), & \xi<\rho\end{Bmatrix}$$
$$-2\alpha^2\rho\left\{\frac{(1+\nu)(b/a)^2}{2[(1-\nu)+(1+\nu)(b/a)^2]}\right.$$
$$+\int_0^\infty[\exp(\pi s)-1]^{-1}I_0(\alpha\xi s)sI_0(\alpha\rho s)\mathrm{d}s$$
$$\left.-\frac{1}{2}\sum_{m=2,4,\ldots}^{\infty}L_mJ_0(m\alpha\xi)J_0(m\alpha\rho)\right\} \tag{35}$$

$$f(\rho)=\alpha\rho\{(1+3\nu)(b/a)^4-(1-\nu)-4(b/a)^2[\nu+(1+\nu)$$
$$\times(b/a)^2\log(b/a)]\}/\{8[(1-\nu)+(1+\nu)(b/a)^2]\} \tag{36}$$

and

$$F(k)-\int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}\mathrm{d}\theta \tag{37}$$

which is an elliptic integral of the first kind.

Numerical Results

Since the kernel of Eq. (33) is infinite for $\xi=\rho$, a procedure given by Stahl and Keer[6] is used to replace the integral equation with a finite number of algebraic equations as follows:

$$\Psi(\rho)+\int_0^1 K(\rho,\xi)[\Psi(\xi)-\Psi(\rho)]\mathrm{d}\xi$$
$$+\Psi(\rho)\int_0^1 K(\rho,\xi)\mathrm{d}\xi=f(\rho), \qquad 0\leq\rho\leq 1 \tag{38}$$

It is seen that the logarithmic singularity at $\xi=\rho$ in the first integral is eliminated by the factor $[\Psi(\xi)-\Psi(\rho)]$. With the use of identities[10]

$$\int_0^\rho \log\left[1-\left(\frac{\xi}{\rho}\right)^2\right]^{1/2}\mathrm{d}\xi=\rho(\log 2-1) \tag{39}$$

and

$$\int_\rho^1\left(\frac{\rho}{\xi}\right)\log\left[1-\left(\frac{\rho}{\xi}\right)^2\right]^{1/2}\mathrm{d}\xi$$
$$=\rho\int_0^{(1-\rho^2)^{1/2}}\frac{u\log u\mathrm{d}u}{(1-u^2)}, \qquad \rho<1 \tag{40}$$

the second integral of Eq. (38) becomes

$$\int_0^1 K(\rho,\xi)\mathrm{d}\xi=\int_0^1$$
$$\times\left\{K(\rho,\xi)+\begin{matrix}\dfrac{(1+\nu)}{\pi}\alpha\left(\dfrac{\rho}{\xi}\right)\log\left[1-\left(\dfrac{\rho}{\xi}\right)^2\right]^{1/2}, & \xi>\rho\\ \dfrac{(1+\nu)}{\pi}\alpha\log\left[1-\left(\dfrac{\xi}{\rho}\right)^2\right]^{1/2}, & \xi<\rho\end{matrix}\right\}\mathrm{d}\xi$$
$$-\frac{(1+\nu)}{\pi}\alpha\rho\left[(\log 2-1)+\int_0^{(1-\rho^2)^{1/2}}\frac{u\log u\mathrm{d}u}{(1-u^2)}\right] \tag{41}$$

Simpson's rule was used to transform the integral equation to simultaneous algebraic equations for determining $\Psi(\rho)$. Eleven equations were used for the calculation.

The deflection in Eq. (18) can be written in the form

$$w=\frac{qa^4}{D}\left\{\frac{(r/a)^4-1}{64}-\frac{4(1+\nu)\log(b/a)+(3+\nu)}{16(1-\nu)}\right.$$
$$\times\left(\frac{b}{a}\right)^4\log\left(\frac{r}{a}\right)-\frac{B_0D}{qa^2}\left[1-\left(\frac{r}{a}\right)^2-2\left(\frac{b}{a}\right)^2\right.$$
$$\left.\times\frac{(1+\nu)}{(1-\nu)}\log\left(\frac{r}{a}\right)\right]-\frac{(b/a)^2(r/a)^2\log(r/a)}{8}$$
$$+\alpha\int_0^1\Psi(\rho)\sum_{m=2,4,\ldots}^{\infty}\left[h_m\left(\frac{r}{a}\right)^m+\left(\frac{r}{a}\right)^{m+2}\right.$$
$$-\frac{(m-1)(1-\nu)}{(3+\nu)}\left(\frac{b}{a}\right)^{2m}\left(\frac{r}{a}\right)^{-m}h_m$$
$$-\frac{(m^2-1)(1-\nu)^2+(3+\nu)^2}{m(3+\nu)(1-\nu)}\left(\frac{b}{a}\right)^{2(m+1)}\left(\frac{r}{a}\right)^{-m}$$
$$+\frac{m(1-\nu)}{(3+\nu)}\left(\frac{b}{a}\right)^{2(m-1)}\left(\frac{r}{a}\right)^{(2-m)}h_m$$
$$\left.+\frac{(m+1)(1-\nu)}{(3+\nu)}\left(\frac{b}{a}\right)^{2m}\left(\frac{r}{a}\right)^{(2-m)}\right]$$
$$\left.\times\frac{J_0(m\alpha\rho)\cos m\theta\mathrm{d}\rho}{G_m}\right\} \tag{42}$$

in which h_m, G_m, and B_0 are defined by Eqs. (17), (21), and (27), respectively. It is noted that the constant D in the third term on the right-hand side is cancelled upon the substitution of B_0.

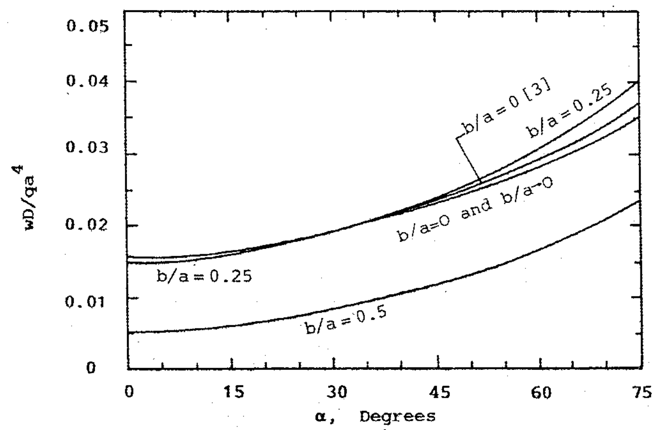

Fig. 2 Deflections at $\theta=0$ deg ($r=b$ and $\nu=0.30$).

The radial moment and the circumferential moment are defined as

$$
\begin{aligned}
M_r = -qa^2 \Bigg\{ & \frac{(3+\nu)}{16}\left(\frac{r}{a}\right)^2 + \left[4(1+\nu)\log\left(\frac{b}{a}\right) + (3+\nu)\right] \\
& \times \frac{(b/a)^4 (r/a)^{-2}}{16} - \frac{2B_0 D}{qa^2}(1+\nu)\left[\left(\frac{b}{a}\right)^2\left(\frac{r}{a}\right)^{-2} - 1\right] \\
& - \left[2(1+\nu)\log\left(\frac{r}{a}\right) + (3+\nu)\right]\frac{(b/a)^2}{8} \\
& + \alpha\int_0^1 \Psi(\rho) \sum_{m=2,4,\ldots}^{\infty} \Bigg[m(m-1)(1-\nu)h_m\left(\frac{r}{a}\right)^{(m-2)} \\
& + (m+1)[(m+2)-\nu(m-2)]\left(\frac{r}{a}\right)^m \\
& - \frac{m(m^2-1)(1-\nu)^2}{(3+\nu)} h_m \left(\frac{b}{a}\right)^{2m}\left(\frac{r}{a}\right)^{-(m+2)} \\
& - \frac{(m+1)[(m^2-1)(1-\nu)^2 + (3+\nu)^2]}{(3+\nu)}\left(\frac{b}{a}\right)^{2(m+1)} \\
& \times \left(\frac{r}{a}\right)^{-(m+2)} - \frac{m(m-1)(1-\nu)[(2-m)+\nu(2+m)]}{(3+\nu)} \\
& \times h_m\left(\frac{b}{a}\right)^{2(m-1)}\left(\frac{r}{a}\right)^{-m} \\
& - \frac{(m^2-1)(1-\nu)[(2-m)+\nu(2+m)]}{(3+\nu)} \\
& \times \left(\frac{b}{a}\right)^{2m}\left(\frac{r}{a}\right)^{-m}\Bigg] \frac{J_0(m\alpha\rho)\cos m\theta \, d\rho}{G_m}\Bigg\}
\end{aligned}
\tag{43}
$$

and

$$
\begin{aligned}
M_\theta = -qa^2 \Bigg\{ & \frac{(1+3\nu)}{16}\left(\frac{r}{a}\right)^2 + \left[4(1+\nu)\log\left(\frac{b}{a}\right) + (3+\nu)\right] \\
& \times \frac{(b/a)^4 (r/a)^{-2}}{16} + \frac{2B_0 D}{qa^2}(1+\nu)\left[\left(\frac{b}{a}\right)^2\left(\frac{r}{a}\right)^{-2} + 1\right] \\
& - \left[2(1+\nu)\log\left(\frac{r}{a}\right) + (1+3\nu)\right]\frac{(b/a)^2}{8} \\
& + \alpha\int_0^1 \Psi(\rho) \sum_{m=2,4,\ldots}^{\infty} \Bigg[m(1-m)(1-\nu)h_m\left(\frac{r}{a}\right)^{(m-2)} \\
& + (m+1)[(2-m)+\nu(m+2)]\left(\frac{r}{a}\right)^m \\
& + \frac{m(m^2-1)(1-\nu)^2}{(3+\nu)} h_m \left(\frac{b}{a}\right)^{2m}\left(\frac{r}{a}\right)^{-(m+2)} \\
& + \frac{(m+1)[(m^2-1)(1-\nu)^2 + (3+\nu)^2]}{(3+\nu)}\left(\frac{b}{a}\right)^{2(m+1)} \\
& \times \left(\frac{r}{a}\right)^{-(m+2)} - \frac{m(m-1)(1-\nu)[(2+m)+\nu(2-m)]}{(3+\nu)} \\
& \times h_m\left(\frac{b}{a}\right)^{2(m-1)}\left(\frac{r}{a}\right)^{-m} \\
& - \frac{(m^2-1)(1-\nu)[(2+m)+\nu(2-m)]}{(3+\nu)} \\
& \times \left(\frac{b}{a}\right)^{2m}\left(\frac{r}{a}\right)^{-m}\Bigg] \frac{J_0(m\alpha\rho)\cos m\theta \, d\rho}{G_m}\Bigg\}
\end{aligned}
\tag{44}
$$

Again, h_m, G_m, and B_0 are defined by Eqs. (17), (21), and (27), respectively.

It is of interest to consider the following two limiting cases:

1) When $b=0$, Eqs. (42–44) become

$$w = -B_0 a^2 - a^4 q/64D \tag{45}$$

$$M_r = -2D[(1+\nu)B_0 + (\nu-1)a^2 B_2] \tag{46}$$

$$M_\theta = -2D[(1+\nu)B_0 + (1-\nu)a^2 B_2] \tag{47}$$

Equations (45–47) correspond to the deflection, radial moment, and circumferential moment for the case of a circular plate as in Ref. 6.

2) When $r=b$ and $b \rightarrow 0$, Eqs. (42–44) become

$$w = -B_0 a^2 - a^4 q/64D \tag{48}$$

$$M_r = 0 \tag{49}$$

$$M_\theta = -4D\left[(1+\nu)B_0 + \frac{(1+\nu)^2}{(3+\nu)}a^2 B_2\right] \tag{50}$$

In this case, the annular plate becomes a circular plate with a small hole at the center and the moment concentration can be determined from Eq. (50).

It is seen that the deflections of both cases are the same, but the circumferential moments are different.

The unknown function $\Psi(\rho)$ is difficult to calculate when $\alpha > 75$ deg and $b/a > 0.5$. Therefore, the numerical results are limited only in the cases $0 \leq \alpha \leq 75$ deg and $0 \leq b/a \leq 0.5$.

The deflections of annular plates at $\theta = 0$ deg and $r=b$ are shown in Fig. 2. It is seen that, with the increase in the simply supported length α, the deflections are increasing. The deflections are not much different when b/a is between 0 to 0.25 but they are significantly reduced when $b/a = 0.5$.

Figure 3 shows the circumferential moments of annular plates at $\theta = 0$ deg and $r=b$. Comparing the cases $b/a = 0$ and $b/a \rightarrow 0$, it is seen that the moment concentrations are approximately equal to 2 for all values of α. In the cases of $b/a = 0.25$ and 0.5, the circumferential moments as well as their differences are both small.

The radial moments at $\theta = 0$ deg for various ratios of b/a and $\alpha = 0$ and 75 deg are shown in Fig. 4. It is noted that, in the case of $\alpha = 0$ deg, the plate corresponds to the case of an annular plate with a clamped edge.

The numerical values of Figs. 2–4 are tabulated in Tables 1–3, respectively.

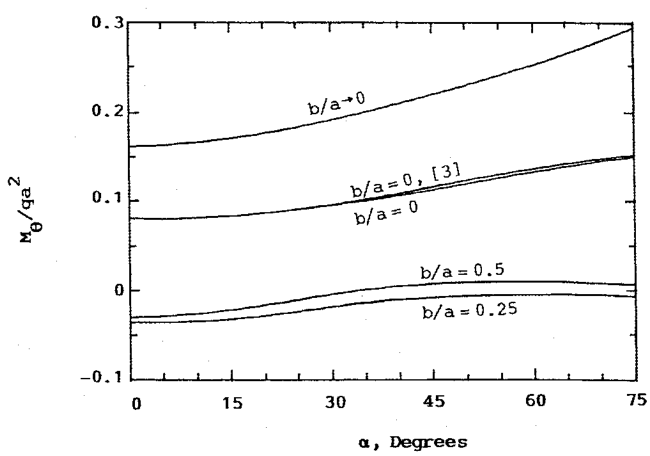

Fig. 3 Circumferential moments at $\theta = 0$ deg ($r=b$ and $\nu = 0.30$).

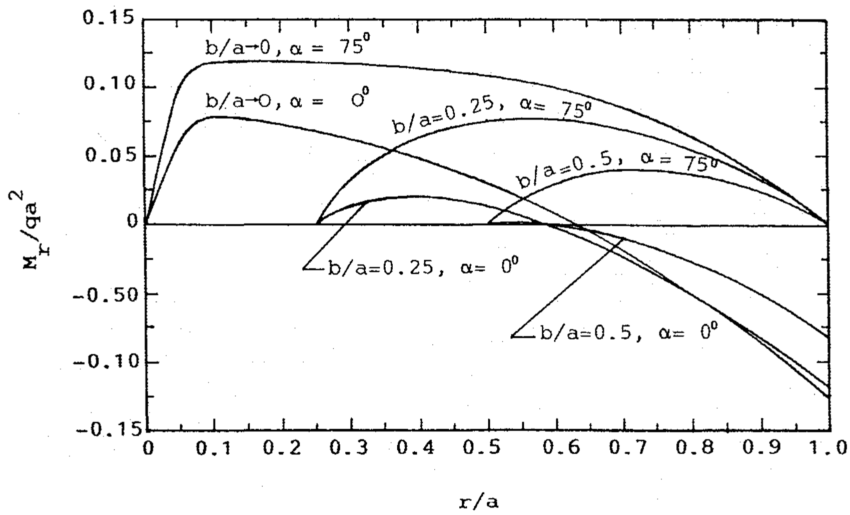

Fig. 4 Radial moments at $\theta = 0$ deg ($\nu = 0.30$).

Table 1 Deflections of plates (wD/qa^2 at $\theta = 0$ deg, $r = b$, and $\nu = 0.30$)

α, deg	$b/a \to 0$ and $b/a = 0^a$	$b/a = 0^b$	$b/a = 0.25$	$b/a = 0.50$
0	0.0156	0.0156	0.0150	0.0053
15	0.0166	0.0166	0.0160	0.0061
30	0.0191	0.0192	0.0191	0.0084
45	0.0229	0.0235	0.0239	0.0118
60	0.0281	0.0288	0.0306	0.0166
75	0.0353	0.0366	0.0401	0.0237

[a]From Ref. 6. [b]From Ref. 3.

Table 2 Circumferential moments (M_θ/qa^2 at $\theta = 0$ deg, $r = b$, and $\nu = 0.30$)

α, deg	$b/a = 0^a$	$b/a = 0^b$	$b/a \to 0$	$b/a = 0.25$	$b/a = 0.50$
0	0.0812	0.0812	0.1624	−0.0360	−0.0314
15	0.0863	0.0864	0.1704	−0.0305	−0.0202
30	0.0986	0.0995	0.1910	−0.0192	−0.0028
45	0.1147	0.1175	0.2200	−0.0094	0.0070
60	0.1321	0.1351	0.2547	−0.0052	0.0093
75	0.1552	0.1534	0.2950	−0.0076	0.0059

[a]From Ref. 6. [b]From Ref. 3.

Table 3 Radial moments (M_r/qa^2 at $\theta = 0$ deg and $\nu = 0.30$)

	$b/a = 0$			$b/a = 0.25$			$b/a = 0.5$	
r/a	$\alpha = 0$ deg	$\alpha = 75$ deg	r/a	$\alpha = 0$ deg	$\alpha = 75$ deg	r/a	$\alpha = 0$ deg	$\alpha = 75$ deg
0	0	0	0.250	0	0	0.50	0	0
0.1	0.0792	0.1177	0.325	0.0169	0.0429	0.55	0.0012	0.0182
0.2	0.0729	0.1176	0.400	0.0199	0.0636	0.60	−0.0004	0.0299
0.3	0.0626	0.1165	0.475	0.0155	0.0738	0.65	−0.0044	0.0368
0.4	0.0481	0.1137	0.550	0.0061	0.0775	0.70	−0.0103	0.0399
0.5	0.0295	0.1079	0.625	−0.0072	0.0759	0.75	−0.0180	0.0398
0.6	0.0068	0.0981	0.700	−0.0237	0.0698	0.80	−0.0276	0.0369
0.7	−0.0200	0.0832	0.775	−0.0433	0.0596	0.85	−0.0385	0.0318
0.8	−0.0511	0.0630	0.850	−0.0657	0.0456	0.90	−0.0511	0.0247
0.9	−0.0851	0.0375	0.925	−0.0907	0.0282	0.95	−0.0649	0.0101
1.0	−0.1250	0	1.000	−0.1185	0	1.00	−0.0805	0

References

[1]Nowacki, W. and Olesiak, Z., "Vibration, Buckling and Bending of a Circular Plate Clamped along Part of Its Periphery and Simply Supported on the Remaining Part," Academie Polonaise des Sciences, Paris, Bulletin C1.IV (also, *Applied Mechanics*, Vol. 4, No. 4, 1956, pp. 247-258).

[2]Conway, H. D. and Farnham, K. A., "Deflections of Uniformly Loaded Circular Plates with Combinations of Clamped, Simply Supported and Free Boundary Conditions," *International Journal of Mechanical Sciences*, Vol. 9, 1967, pp. 661-671.

[3]Leissa, A. W. and Clausen, W. E., "Deflection of a Circular Plate Having Mixed Boundary Conditions," *AIAA Journal*, Vol. 5, Dec. 1967, pp. 2287-2288.

[4]Bartlett, C. C., "The Vibration and Buckling of a Circular Plate Clamped on Part of Its Boundary and Simply Supported of the Remainder," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 16, 1963, pp. 431-440.

[5]Noble, B., "The Vibration and Buckling of a Circular Plate Clamped on Part of Its Boundary and Simply Supported on the Remainder," *Developments in Mechanics*, Vol. 3, Pt. 2, "Dynamics and Fluid Mechanics," Proceedings of Ninth Midwestern Mechanics Conference, April 1965, Wiley, New York, 1967, pp. 141-146.

[6]Stahl, B. and Keer, L. M., "Bending of a Uniformly Loaded Circular Plate with Mixed Boundary Conditions," *AIAA Journal*, Vol. 10, 1972, pp. 739-743.

[7]Keer, L. M., Dundurs, J., and Kiattikomol, K., "Separation of a Smooth Circular Inclusion from a Matrix," *International Journal of Engineering Sciences*, Vol. 11, No. 11, pp. 1221-1231.

[8]Williams, M. L., "Surface Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates Under Bending," *Proceedings of First U. S. National Congress of Applied Mechanics*, ASME, New York, 1952, pp. 325-329.

[9]Timoshenko, S. and Woinowsky-Krieger, S., *Theory of Plates and Shells*, 2nd ed., McGraw-Hill, New York, 1959.

[10]Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals Series and Products*, 4th ed., Academic Press, New York, 1956, pp. 427, 584.

[11]Dwight, H. B., *Table of Integrals and Other Mathematical Data*, 4th ed., Macmillan, New York, 1969, pp. 179-186.